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# $\zeta$- and $\eta$-function resummation of infinite series: general case

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**Abstract.** Given a spectrum of positive numbers $\{\lambda_m\}$ from which a $\zeta$-function $Z(s)=\sum_m \lambda_m^{-s}$ can be constructed, the reorganization of series of the type

$$F(s,t)\equiv\sum_m \lambda_m^{-s} f(\lambda_m t)$$

into power series in $t$ is examined in detail using the method of $\zeta$-function resummation. For summand functions $f(\lambda_m t)$ having power series expansions in $\lambda_m t$ with infinite radius of convergence, and which satisy other conditions of a rather general nature, we find that $F(s,t)$ can be reorganized to

$$F(s,t)=\sum_n a_n t^{b_n}+\sum_k c_k t^{d_k}\ln t+R(s,t)$$

where $R(s,t)$ vanishes exponentially as $t\to 0$. The numbers $a_n$, $b_n$, $c_n$, $d_n$ can all be computed in terms of the $\zeta$-function $Z(s)$. $R(s,t)$ is difficult to evaluate, but important general features of this function can be determined. The power series expansion of $F(s,t)$ can be regarded as a generalization of the heat kernel expansion (for which $f(\lambda_m t)=\exp(-\lambda_m t)$ and $s=0$) to non-zero complex variable $s$ (which is useful) and to many other summand functions $f(\lambda_m t)$. Remarkably, the $\zeta$-function resummation method can be applied as easily to *divergent* series $F(s,t)$ as it can to *convergent* ones. The method is therefore both a rearrangement procedure for convergent series, and a summation prescription for divergent series.

## 1. Introduction

The problem addressed in this paper is that of expanding functions defined by a series

$$F(s,t)\equiv\sum_m \lambda_m^{-s} f(\lambda_m t) \tag{1.1}$$

in powers of the real parameter $t$. Here $s$ is a complex variable, and $f(z)$ is a transcendental function whose power series expansion

$$f(z)=\sum_{k=0}^{\infty}(-)^k C(k) z^{ak+b} \qquad a>0 \qquad b\geqslant 0 \qquad |z|<\infty \tag{1.2}$$

has infinite radius of convergence. For much of our discussion the power series (1.2) will be assumed to have alternating sign as indicated—i.e. $C(k)$ does not alternate in sign. Another assumption made about $C(z)$ (now regarded as a function in the complex plane) is that $C(z)$ is regular for $\operatorname{Re} z>-\infty$. Moreover, the points $z=-n$, $n=1$, 2, 3,... are assumed to be zeros of $C(z)$. The spectrum $\{\lambda_m\}$ might be regarded as belonging to a real, positive operator. We prefer, however, not to link the real, positive spectrum $\{\lambda_m\}$ to an operator, but rather to think of this spectrum in a less specific

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way, as any sequence growing without limit, having no infinite degeneracies, and generally suitable for constructing a $\zeta$-function

$$Z(s) \equiv \sum_m \lambda_m^{-s} \qquad \operatorname{Re} s > B \tag{1.3}$$

where $\operatorname{Re} s = B$ is the abscissa of absolute convergence of $Z(s)$.

The general solution of the problem just posed will be shown to have the form

$$F(s, t) = \sum (s, t) + \{s, t\}_{\mathrm{p}} + \{s, t\}_{\mathrm{ex}} \tag{1.4}$$

where each of the three functions on the right has a highly distinctive character. The function

$$\sum (s, t) = \sum_{k=0}^{\infty} (-)^k C(k) t^{ak+b} Z(s - ak - b) \tag{1.5}$$

is a power series involving constant powers of $t$. The function $\{s, t\}_{\mathrm{p}}$ represents another power series in $t$, having $s$-dependent powers. Also, for special values of $s$, this function contains $\ln t$ terms. We shall compute $\{s, t\}_{\mathrm{p}}$ explicitly in terms of $Z(s)$. The remaining function $\{s, t\}_{\mathrm{ex}}$ has no power or logarithmic dependence on $t$; it is a function of $t$ beyond power series form, akin to $\exp(-1/t)$ in that it vanishes with $t$ faster than any power of $t$. This property of $\{s, t\}_{\mathrm{ex}}$ is very important, for it is extremely difficult to compute $\{s, t\}_{\mathrm{ex}}$ exactly. Equation (1.4) is understood to be exact, but one cannot, as a rule, do more than estimate $\{s, t\}_{\mathrm{ex}}$. Equation (1.4) is nevertheless useful because, for small $t$, one can discard $\{s, t\}_{\mathrm{ex}}$ relative to the power series, leaving the quite general asymptotic series formula

$$F(s, t) \sim \sum (s, t) + \{s, t\}_{\mathrm{p}} \qquad t \to 0+ \tag{1.6}$$

in which everything on the right is known in terms of $Z(s)$, the central function in these considerations. If one can calculate $Z(s)$, say in terms of simpler $\zeta$-functions whose properties are well understood, then equation (1.6) is a quite explicit as well as general formula giving $F(s, t)$ as an asymptotic series in $t$. Beyond this, one can hope that equation (1.4) may eventually be used in a more precise fashion by learning more about the small function $\{s, t\}_{\mathrm{ex}}$. This will be a subject for future research.

Equations (1.1)-(1.6) were discussed in simplified form (with $\lambda_m = m^\alpha$, $m = 1, 2, 3, \ldots$ and $\alpha > 0$) in [1]. That paper draws upon and augments work in a number of earlier papers [2-7] on the use of $\zeta$-function regularization to rearrange infinite series of the form $\Sigma_1^\infty m^{-\alpha s} f(tm^\alpha)$ into power series in $t$. Nowhere, to the author's knowledge, has the extension of these methods from the integers $\{m\}$ to an arbitrary spectrum been given. The main purpose of the present article is to work through the details of this extension.

Equation (1.6) reproduces well-known results on the power series expansion of the heat kernel $\Sigma_m \exp(-\lambda_m t)$, and generalizes these to the non-zero complex variable $s$, and to summand functions $f(\lambda_m t)$ different from the exponential $f(\lambda_m t) = \exp(-\lambda_m t)$. There is a practical value in introducing the complex variable $s$, beyond mere generalization. Analyticity considerations in $s$ reveal a close relationship between the two power functions $\Sigma\,(s, t)$ and $\{s, t\}_{\mathrm{p}}$ in equations (1.4), (1.6). Our derivation of equation (1.4) involves an 'unjustified' series commutation which generates an additional *a priori* unknown function, divided in equation (1.4) into the two terms $\{s, t\}_{\mathrm{p}}$ and $\{s, t\}_{\mathrm{ex}}$. This separation is unique and unambiguous: $\{s, t\}_{\mathrm{p}}$ is a power series in $t$, while $\{s, t\}_{\mathrm{ex}}$

vanishes exponentially with $t$. The power function $\{s, t\}_p$ is constrained by the following consideration. $F(s, t)$ in equation (1.1) is the $\zeta$-function series (1.3) with an additional factor $f(\lambda_m t)$ inserted into the summand. The condition that $F(s, t > 0)$ does *not* have the poles of $Z(s)$—these having been prevented from forming by the disordering effect of the summand factor $f(\lambda_m t)$—can only be satisfied if the power function $\{s, t\}_p$ cancels the poles in $s$ found in the other power function $\Sigma(s, t)$. (In equation (1.5) these $s$-poles are in the coefficients $Z(s-ak-b)$.) The exponentially small function $\{s, t\}_{ex}$ cannot participate in the pole cancellation because it has the wrong $t$ dependence. Consequently, the meromorphic structure of $\Sigma(s, t)$ uniquely determines the meromorphic structure of $\{s, t\}_p$, and this function itself up to an entire function of $s$. There is a systematic procedure [4, 7] for determinig the correct entire function, which simultaneously provides a contour integral formula for the exponentially small function $\{s, t\}_{ex}$ (see the appendix). The latter function is an entire function of $s$—related, of course, to the other terms $\Sigma(s, t)$ and $\{s, t\}_p$ in equation (1.4), but at a far deeper level than $\Sigma$ and $\{\}_p$ are related to one another.

The general results in this paper can be used at two quite different levels:

(i) Rearrangement of convergent series. When the series (1.1) converges in some half-plane $\mathrm{Re}\, s > C$, equations (1.1)–(1.6) describe a series rearrangement problem in which $\zeta$-function regularization of the coefficients in equation (1.5) is the distinguishing feature of the method. Typical of this category is the heat kernel problem (see section 4).

(ii) Summation of divergent series. The series (1.1) may also diverge, perhaps very strongly. Then one has to reinterpret the series commutation procedure, and yet the same steps as before again yield unambiguous results. $\zeta$-function resummation has now become a summation prescription for assigning a unique power series representation plus exponentially small correction to the original divergent series. This procedure does not seem to be part of the standard methodology for divergent series (see [8]).

Equations (1.1)–(1.6) apply to a broad range of series of interest for theoretical physics and for applied mathematics. The spectrum $\{\lambda_m\}$ could be the energy spectrum of a quantum system, for example. Or, in quantum field theory, this could be the spectrum of the Lagrangian kinetic operator $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$ on a curved spacetime of interest. We mention that $\zeta$-function resummation can be used at the local level. To make the idea clear, let $\psi_m(x) = \langle x | m\rangle$ be the eigenfunctions $A\psi_m(x) = \lambda_m\psi_m(x)$ of operator $A$. Replacing the series (1.1) by $F(s, t | x, y) = \langle x | A^{-s} f(At) | y\rangle$ and $Z(s)$ by $Z(s | x, y) = \langle x | A^{-s} | y\rangle$, one readily obtains the local version of equation (1.4). This will be discussed separately and in detail [15].

## 2. $\zeta$-function resummation

$\zeta$-function resummation involves 'non-allowed' series commutation: a divergent series is commuted through a convergent summation. This may, in general, result in an unknown function being generated, whose determination is the main problem to be solved. We separate this unknown function $\{\}$ into two parts, $\{\}_p + \{\}_{ex}$, agreeing beforehand that $\{\}_p$ shall contain all power dependence on $t$ (plus $\ln t$ dependence when present), while $\{\}_{ex}$ contains all terms vanishing exponentially with $t$. This separation is clearly unique and unambiguous within the elementary types of $t$ dependence listed (these do suffice). We shall compute $\{\}_p$ explicitly. Moreover, certain general properties of $\{\}_{ex}$ will be established; e.g. this function is entire in $s$. The calculated function $\{s, t>0\}_p$ is meromorphic, and its poles precisely cancel the poles of the (meromorphic) power function $\Sigma(s, t>0)$ in equation (1.4). As a result, $F(s, t>0)$ is entire in $s$. For the defining series (1.1), the interpretation of this result is that the summand function $f(\lambda_m t)$ disrupts the spectral sum $\Sigma_m$ sufficiently to prevent the poles of $Z(s)$ from forming. However, these poles abruptly reappear in the limit $t \to 0$: $F(s, 0) = f(0)Z(s)$. One can turn the argument around, assume that $F(s, t>0)$ is an entire function of $s$, and deduce from equation (1.4) the meromorphic structure of $\{s, t\}_p$, as mentioned in the introduction. Analyticity in $s$ therefore plays an important role, and we begin with this aspect of the problem.

### 2.1. Analyticity in the s-plane

Often $\zeta$-functions are associated with positive operators $A$ defined on some compact spacetime manifold $\mathcal{M}$. Under rather general conditions (e.g. see [9]), the abscissa of absolute convergence in equation (1.3) is $B = N/d$, where $d$ is the order of the operator $A$ and $N$ is the dimension of spacetime $\mathcal{M}$. Continuation of $Z(s)$ to the left of $\operatorname{Re} s = B$ shows that $Z(s)$ is meromorphic, with a set of regularly spaced poles at points $s = (N-k)/d$, $k = 0, 1, 2, \ldots$ along the real axis, ending at the rightmost pole at $s = N/d$ (which is always on the abscissa of convergence). Some, and perhaps infinitely many, of the pole residues may vanish. $\zeta$-functions never have a pole at $s = 0$.

The series (1.1) is the numerical series (1.3) which defines $Z(s)$, modified by having an extra factor $f(\lambda_m t)$ inserted into the summand. Obviously, for $t = 0$ this factor has no effect. However, as long as $t > 0$, so the summand factor $f(\lambda_m t)$ depends non-trivially on the eigenvalues $\lambda_m$, this extra factor will disrupt the subtle and delicate process of pole formation in $Z(s)$. There are many known explicit examples of this.

One example of the suppression of $\zeta$-function pole formation is the exponential series

$$F(s, t) = \sum_m \lambda_m^{-s} \, e^{-\lambda_m t}$$

which converges for $\operatorname{Re} s > -\infty$ due to the exponential damping of large eigenvalues. At $t = 0$ the abscissa of convergence jumps from $\operatorname{Re} s = -\infty$ back to $\operatorname{Re} s = B$, and the poles of $Z(s)$ are restored. A different example is the Epstein function [10]

$$F(s, h_a) = \sum_{m_a=-\infty}^{\infty} (a_1 m_1^2 + \ldots + a_N m_N^2)^{-s} \, e^{2\pi i(m_1 h_1 + \ldots + m_N h_N)}$$

which defines an entire function of $s$ as long as not all the constants $h_a$ $(a = 1, 2, \ldots, N)$ are integers. Here we see that pure phase factors are sufficiently disruptive to destroy $\zeta$-function pole formation. For $h_a$ = integer for all $a$, $F(s, h_a)$ becomes a $\zeta$-function with a single pole at $s = N/2$ [10]. But this pole is eliminated when even one of the constants $h_a$ becomes non-integral.

### 2.2. Series commutation

$\zeta$-function resummation applied to equation (1.1) means the following:

$$\begin{aligned} F(s, t) &\equiv \sum_m \lambda_m^{-s} f(\lambda_m t) \\ &= \sum_m \lambda_m^{-s} \sum_{k=0}^{\infty} (-)^k C(k) (\lambda_m t)^{ak+b} \\ &= \sum (s, t) + \{(2.1)\} \end{aligned} \tag{2.1}$$

where $\Sigma(s, t)$ is the series (1.5) and $\{(2.1)\}$ represents the function generated by commuting the summations $\Sigma_m$ and $\Sigma_k$ of the first equality. Let us postpone discussion of this function (i.e. of the series commutation problem) and first look at $\Sigma(s, t)$.

The coefficients in this series are, up to elementary factors, just the $\zeta$-functions $Z(s-ak-b)$. It is important that, for the infinitely many terms having $k >$ $(\mathrm{Re}\, s - B - b)/a$, $Z(s-ak-b)$ is being evaluated to the left of its abscissa of convergence. Assuming for the moment that the point $s-ak-b$ is not a pole of the $\zeta$-function for any $k = 0, 1, 2, \ldots$, it follows that every coefficient

$$\sum_m \lambda_m^{-s+ak+b} = Z(s-ak-b) \tag{2.2}$$

in equation (1.5) is *finite*. Analytic continuation is being used to assign, term by term, a *unique* finite value to the (divergent, for $\mathrm{Re}\, s < B + ak + b$) series (2.2). Infinitely many divergent coefficients in the power series (1.5) are being simultaneously regularized in this way. The power series $\Sigma(s, t)$ is amazingly easy to obtain by this method, and all complications reside in the 'extra term' as we shall call it— the *a priori* unknown function $\{(2.1)\}$ generated by series commutation.

*The extra term $\{(2.1)\}$ arises because a divergent sum ($\Sigma_m$ in equation (2.2)) is* being commutated through a convergent sum ($\Sigma_k$). Initially, all one can say is that an extra function probably is generated. The series commutation problem is to find out what this function is. One can solve this problem at a quite general level, up to exponentially small terms whose full evaluation remains a problem for future research. This is done by means of a Cauchy integral argument, initiated by Weldon [4] and improved by Elizalde and Romeo [7].

As previously mentioned, we divide the extra term $\{(2.1)\}$ generated by series commutation into two parts,

$$\{(2.1)\} = \{(2.1)\}_{\mathrm{p}} + \{(2.1)\}_{\mathrm{ex}}. \tag{2.3}$$

The subscripts p (and ex) refer to the power (and exponentially small) $t$-behaviour of these functions. Weldon's argument [4] gives us the power function $\{\,\}_{\mathrm{p}}$ explicitly (see the appendix). As mentioned previously, this function $\{\,\}_{\mathrm{p}}$ has poles in $s$ which cancel all the $s$-poles of the power series $\Sigma(s, t)$ in equation (1.4). Elizalde and Romeo [7] pointed out the existence of the additional term $\{\,\}_{\mathrm{ex}}$ in equation (2.3), but did not discuss this function's most inportant properties [1]: $\{\,\}_{\mathrm{ex}}$ vanishes with $t$ faster than any power of $t$, and $\{\,\}_{\mathrm{ex}}$ is an entire function of $s$ (see the appendix). The two terms on the right in equation (2.3) play clear and highly distinctive roles in the mathematics of $\zeta$-function resummation. Only a full appreciation of these roles will enable one to understand the method in its entirety.

Before writing down $\{\,\}_{\mathrm{p}}$ we must introduce some notation. The poles of $Z(s)$ are denoted by $s = B - \Delta_n$, $n = 0, 1, 2, \ldots$ with $\Delta_0 = 0$ and positive pole spacings $\Delta_n$ for $n = 1, 2, \ldots$. Near these poles

$$Z(B - \Delta_n + \varepsilon) = \frac{1}{\varepsilon} R_n + C_n + \mathrm{O}(\varepsilon) \tag{2.4}$$

where $R_n$ and $C_n$ are constants.

Recalling that we have assumed $C(z)$ to be regular for $\mathrm{Re}\, s > -\infty$, one readily sees *that all poles cancel in equation* (2.1) if

$$\{\,\}_{\mathrm{p}} = \sum_{n=0}^{\infty} \left(-\frac{\pi}{a}\right) \operatorname{cosec} \frac{\pi}{a}(s - B - b + \Delta_n) R_n C\left(\frac{1}{a}(s - B - b + \Delta_n)\right) t^{s-B+\Delta_n}. \tag{2.5a}$$

Here cosec $\pi(s-B-b+\Delta_n)/a$ has poles at $s=ap+b+B-\Delta_n$, $p=0, \pm 1, \pm 2, \ldots$. Of these, the poles labelled by $p=0, 1, 2, \ldots$ cancel the poles in $Z(s-ak-b)$, $k=0, 1, 2, \ldots$. There remain the poles in cosec labelled by $p=-1, -2, \ldots$. These latter, at the points $s=ap+b+B-\Delta_n$, are eliminated because the coefficients multiplying them are strictly zero:

$$C\left(\frac{1}{a}(s-B-b+\Delta_n)\right)=C(p)$$

and $C(p)=0$ for $p=-1, -2, \ldots$ by assumption in equation (1.2). Therefore, the extra term (2.5$a$) has only the poles needed to cancel the $\zeta$-function poles in equation (2.1), and $F(s, t>0)$ is indeed represented by this formula as being regular for Re $s>-\infty$. Equation (2.5$a$) is *derived* in the appendix by the Cauchy integral method for dealing with series commutation.

Equations (2.1), (2.5$a$) are valid for all finite $s$. For points $s$ where pole cancellation is involved, the formula specifically implementing this pole cancellation should be given:

$$\{\,\}_{\mathrm{P}}=\sum_n{}'\left(-\frac{\pi}{a}\right)\operatorname{cosec}\frac{\pi}{a}(s-B-b+\Delta_n)R_nC\left(\frac{1}{a}(s-B-b+\Delta_n)\right)t^{s-B+\Delta_n}$$
$$+\sum_n{}''(-)^L t^{aL+b}\left(C(L)(C_n-\ln t)-\frac{1}{a}C'(L)\right) \tag{2.5$b$}$$

where $\Sigma'$ ($\Sigma''$) is a sum over all poles for which $(s-B-b+\Delta_n)/a$ is not (is) equal to any integer $L=0, 1, 2, \ldots$. When equation (2.5$b$) is used, the singular terms in the power series $\Sigma(s, t)$ are of course omitted. Note that the extra term (2.5$b$) contains $\ln t$ dependence, whereas there are only powers of $t$ in equation (2.5$a$). We see that $\ln t$ terms occur *only* for special values of $s$, and can be regarded as belonging to (or at any rate as originating in) the power series itself.

We proceed to the second term in equation (2.3)—the function $\{\,\}_{\mathrm{ex}}$ which is beyond power series form in $t$. A formal expression for this function is derived in the appendix:

$$\{\,\}_{\mathrm{ex}}=\frac{\mathrm{i}}{2}\int_C \mathrm{d}k \operatorname{cosec}\pi k C(k)t^{ak+b}Z(s-ak-b) \tag{2.6}$$

where $C$ is the counterclockwise semicircle drawn at infinity and bounding the right-half $k$-plane. The importance of this function in the series commutation problem was first pointed out by Elizalde and Romeo [7]. For $C(k)$ vanishing reasonably rapidly as the contour $C$ is approached, the integral (2.6) might be expected to vanish. However, it does not vanish in general, the reason being that the $\zeta$-function $Z(s-ak-b)$ diverges more rapidly than exponentially as the contour $C$ is approached. The parameter $a>0$ controls the rate of this divergence, and therefore plays a decisive role in determining whether $\{\,\}_{\mathrm{ex}}$ vanishes or not (see the appendix). The function (2.6) probably cannot be evaluated in terms of known functions. One ought not be surprised by this, considering the ease with which the other part—equation (2.5)—of the series commutation problem has been solved. The series rearrangement problem we are working on is truly non-trivial. Complications beyond those already surmounted are inevitable. These complications manifest themselves in the function (2.6) which contains the really deep mathematical complexity of this problem, and is correspondingly difficult to evaluate.

Given the difficulty of computing the integral (2.6), it is very important that $\{\}_{\text{ex}}$ vanishes with $t$, faster than any power of $t$ [1]. In the appendix we argue this can be interpreted to mean that $\{\}_{\text{ex}}$ vanishes as $t\to 0$ like $O[\exp(-A/t)]$ where $A$ is some constant. This behaviour of $\{\}_{\text{ex}}$ means that, for small $t$, one can neglect $\{\}_{\text{ex}}$ relative to powers of $t$. What remains is an asymptotic series for $F(s, t>0)$.

We now write down our final formula for $F(s, t)$:

$$F(s,t)\equiv\sum_m \lambda_m^{-s} f(\lambda_m t)$$
$$=\sum_{k=0}^{\infty}(-)^k C(k)t^{ak+b}Z(s-ak-b)+\{\}_{\text{p}}+\{\}_{\text{ex}} \tag{2.7}$$

with $\{\}_{\text{p}}$ and $\{\}_{\text{ex}}$ given by equations (2.5) and (2.6) respectively. This formula is understood to be exact if the power series converge. For small $t$ one is free to disregard $\{\}_{\text{ex}}$, and then equation (2.7) becomes the asymptotic series

$$\sum_m \lambda_m^{-s} f(\lambda_m t)\sim\sum_{k=0}^{\infty}(-)^k C(k)t^{ak+b}Z(s-ak-b)+\{\}_p. \tag{2.8}$$

If one has computed the $\zeta$-function $Z(s)$ in terms of known functions, then every term on the right-hand side of this formula is known.

There remains the question of the convergence of the power series in equation (2.7). Actually, there are two power series to consider: the (possibly finite) series (2.5$a$), and the infinite series (1.5). To say much about the power series (2.5$a$) one needs the pole residues $R_n$ in equation (2.4). Lacking useful bounds on these numbers, one can investigate explicit classes of spectra to gain insight of them. This is too involved and specific to present here. The impression one gains from such calculations is that the residues $R_n$ do not diverge strongly as $n\to\infty$, and convergence in equation (2.5$a$) is probably assured by the factor $C[(s-B-b+\Delta_n)/a]$. This point requires more study, however.

We shall be somewhat more specific about the power series (1.5), whose convergence is far from being assured, because of the $\zeta$-function factor $Z(s-ak-b)$ whose argument is going arbitrarily negative as $k\to\infty$. $\zeta$-functions always blow up when this happens, as is briefly discussed in the appendix. If the other factor $C(k)$ cannot tame the divergence of $Z(s-ak-b)$ for $k\to\infty$, then the series (1.5) will not converge for any $t>0$. In practice, any of the following situations might be encountered:

(i) $C(k)\to 0$ as $k\to\infty$ sufficiently strongly that the series (1.5) converges for $t<\infty$;

(ii) $C(k)Z(s-ak-b)\to d^k$ with $d=$ constant, and in this marginal case the series (1.5) converges for $dt^a<1$;

(iii) $C(k)$ fails to provide convergence for any $t>0$.

*Example.* As an illustration we examine the power series

$$\sum_{k=0}^{\infty}(-)^k\frac{1}{\Gamma(ck)}t^{ak+b}\zeta(s-ak-b)$$
$$=\sum_{k=0}^{\infty}(-)^k\frac{1}{\Gamma(ck)}\frac{1}{\pi}(2\pi)^s\left(\frac{t}{2\pi}\right)^{ak+b}\sin\frac{\pi}{2}(s-ak-b)$$
$$\times\Gamma(1-s+ak+b)\zeta(1-s+ak+b)$$

where $\zeta(s)$ is the Riemann $\zeta$-function. The reflection formula (e.g. see [11]) for $\zeta(s-ak-b)$ has been used to exhibit the behaviour of $\zeta(s-ak-b)$ as $k\to\infty$ in terms of more familiar functions. To leading order in $k$ the summand above can be written as

$$(-)^k\frac{1}{\pi}(2\pi)^s\left(\frac{t}{2\pi}\right)^{ak+b}\sin\frac{\pi}{2}(s-ak-b)\frac{\Gamma(ak)}{\Gamma(ck)}$$

where $\zeta(1-s+ak+b)\approx 1$ is used for $k\to\infty$ and finite $s$. The most important factor in the summand is clearly

$$\frac{\Gamma(ak)}{\Gamma(ck)}\propto k^{-(c-a)k} \qquad k\to\infty.$$

When $c<a$, this factor diverges more strongly than exponentially, and the power series diverges for $t>0$ (case (iii)). The case $c=a$ is marginal, with asymptotic summand

$$(-)^k\frac{1}{\pi}(2\pi)^s\,\mathrm{e}^{s-1-b}(ak)^{1+b-s}\left(\frac{t}{2\pi}\right)^{ak+b}\sin\frac{\pi}{2}(s-ak-b)$$

so the power series converges for $t$ sufficiently small (case (ii)). Finally, for $c>a$ the factor $\Gamma(ak)/\Gamma(ck)$ provides convergence stronger than exponential, so the power series converges for $0\leqslant t<\infty$ (case (i)). Other examples with similar features could be given.

### 2.3. A series commutation problem with no extra term

The series commutation problem of section 2.2 will be contrasted here with a different problem in which series commutation is permissible, which means by definition that the extra term vanishes identically.

In equation (1.1), rather than the summand function (1.2) let us use instead

$$g(x)=\sum_{k=0}^{\infty}b_kx^{-k} \qquad x>x_0\geqslant 0 \tag{2.9}$$

where convergence is assumed for the real variable $x>x_0$. Then, with a slight change in notation,

$$\begin{aligned} G(s,t)&\equiv\sum_m\lambda_m^{-s}g(\lambda_m t) && t>x_0/\lambda_1\\ &=\sum_m\lambda_m^{-s}\sum_{k=0}^{\infty}b_k(\lambda_m t)^{-k} && \mathrm{Re}\,s>B\\ &=\sum_{k=0}^{\infty}b_kt^{-k}Z(s+k) && \text{all } s \end{aligned}\tag{2.10}$$

where $\lambda_1$ is the smallest eigenvalue of the spectrum. In the final line we have used

$$Z(s+k)=\sum_m\lambda_m^{-s-k} \qquad \mathrm{Re}(s+k)>B \tag{2.11}$$

to evaluate $\Sigma_m$ for $\mathrm{Re}\,s>B-k$. Then, by $\zeta$-function regularization, the final line in equation (2.10) is valid for all $s$. Because the convergent (for $\mathrm{Re}\,s>B-k$) sum (2.11) is commuted through the convergent sum $\Sigma_k$, no extra term is generated.

The absence of an extra term in equation (2.10) is directly connected with analytic behaviour in $s$ of the function $G(s, t)$. For large eigenvalues $\lambda_m \to \infty$ and $t > 0$, the summand function $g(\lambda_m t)$ has no disordering effect because

$$g(\lambda_m t) \to b_0 = \text{constant} \qquad \lambda_m \to \infty.$$

Consequently, formation of the poles of the $\zeta$-function $Z(s)$ does occur. Indeed, due to the presence of the parameter $t$, a pole structure more complicated than that of $Z(s)$ comes into existence, and this is explicitly revealed by equation (2.10). There is no extra term $\{\,\}_{\rm p}$ needed to cancel the poles of $Z(s+k)$ in equation (2.10), and appropriately $\{\,\}_{\rm p} \equiv 0$ in this formula.

The absence of the other part $\{\,\}_{\rm ex}$ of the extra term (2.3) can also be understood quite easily. The Cauchy integral argument in the appendix leads to a formula much like equation (2.6) for $\{\,\}_{\rm ex}$, with $Z(s+k)$ in place of $Z(s-ak-b)$. Now one of the important things to know about $\zeta$-functions is that they are very tame far to the right in their complex plane. Indeed, for Re $k \gg |\text{Re}\ s|$, the series definition (1.3) of a general $\zeta$-function becomes

$$\begin{aligned} Z(s+k) &= \lambda_1^{-s-k}\left[1+\left(\frac{\lambda_1}{\lambda_2}\right)^{k+s}+\ldots\right] \\ &= \lambda_1^{-s-k} + \text{small corrections} \end{aligned} \tag{2.12}$$

where the smallest eigenvalue $\lambda_1$, the next smallest $\lambda_2$, etc., are assumed non-degenerate for simplicity. A technical point concerning $\zeta$-functions which we can safely ignore everywhere else in this paper is this: when the eigenvalues $\lambda_m$ have physical dimension, it is necessary to employ modified eigenvalues $\lambda_m/\mu$ scaled by a real dimensional constant having the same dimensions as $\lambda_m$. Then the right-hand side of equation (2.12) reads $(\lambda_1/\mu)^{-s-k}+\ldots$ where $\mu$ will determine the numerical value of the ratio $\lambda_1/\mu$. Should this ratio be $<1$, the right-hand side of equation (2.12) is vanishingly small. Even if $\lambda_1/\mu > 1$, there is only a power divergence in equations (2.12) for Re $k \to \infty$. The $\zeta$-function $Z(s-ak-b)$ blows up as Re $k \to \infty$ far more rapidly than this, overwhelming the other integrand factor $C(k)$ in equation (2.6) and causing $\{\,\}_{\rm ex}$ to be non-zero in general. But here, with $Z(s+k)$ either converging or diverging weakly at worst, the other integrand factor $C(k)$ will overwhelm $Z(s+k)$ and $\{\,\}_{\rm ex}$ will vanish.

Equation (2.12) also makes the convergence of the power series (2.10) a trivial matter to settle. For large real $k \gg |\text{Re}\ s|$, the summand in equation (2.10) becomes $\lambda_1^{-s} b_k (t\lambda_1)^{-k}$, and by assumption the series (2.10) converges for $t\lambda_1 > x_0$.

The difference between the functions (2.10) and (2.7)—i.e. between the two series commutation problems—is illustrated by

$$\begin{aligned} G(s,x) &= \sum_{m=1}^{\infty} m^{-s}\, \mathrm{e}^{-x/m} \\ &= \sum_{k=0}^{\infty} (-x)^k \frac{1}{k!} \zeta(s+k) \end{aligned} \tag{2.13}$$

$$\begin{aligned} F(s,x) &= \sum_{m=1}^{\infty} m^{-s}\, \mathrm{e}^{-mx} \\ &= \sum_{k=0}^{\infty} (-x)^k \frac{1}{k!} \zeta(s-k) + \{\Gamma(1-s) x^{s-1}\}_{\rm p}. \end{aligned} \tag{2.14}$$

$F(s, x>0)$ is an entire function of $s$. $G(s, x>0)$ has poles at $s=1-k=1, 0, -1, -2, \ldots$ with residues $(-x)^k/k!$. The power series (2.13) converges for $x<\infty$ or $t\equiv 1/x>0$, because $\zeta(s+k)=1+\ldots$ for real $k \gg |\mathrm{Re}\, s|$. The convergence of the power series (2.14) was discussed in section 2.2 above.

## 3. $\zeta$-function resummation as a divergent series summation prescription

Divergent series may be capable of defining functions. What a divergent series lacks is a convincing regularization prescription which extracts the (finite, well-defined) function from its divergent series. One can hardly expect to succeed in finding such a regularization for all divergent series. However, in many cases a satisfactory result can be achieved. One familiar example is $\zeta$-function regularization, which assigns the finite value $-Z'(0)$ [12], with $Z(s)$ as in equation (1.3), to divergent series of the form $\Sigma_m \ln \lambda_m$. Mathematicians have devised a substantial body of regularization methods, applicable to quite a range of divergent series types. The reader wishing to become familiar with traditional divergent-series summation methods is advised to consult Hardy's classic text [8], both for technical content and for a sense of the various attitudes taken toward this subject.

Our point in this section will be that $\zeta$-function resummation can be applied to divergent series of the form (1.1) just as readily and unambiguously as it can to convergent series. If the series (1.1) is divergent, then $\zeta$-function resummation will reorganize it into a power series in $t$ plus an exponentially small correction. The power series is asymptotic and may diverge—but such a power series divergence is far milder than the divergences one typically begins with. One might speak of 'near regularization' in such cases. The asymptotic series may of course be convergent. Then, one has (modulo the exponentially small term) achieved the complete and unambiguous regularization of the original divergent series. Divergent series which can be dealt with by such methods might well be called '$\zeta$-function summable'.

For the sake of clarity we introduce a minor change in notation. Rather than talking about the series (1.1) let us discuss instead

$$\tilde{F}(s, t) \equiv \sum_m \lambda_m^{-s} \tilde{f}(\lambda_m t) \tag{3.1}$$

where

$$\tilde{f}(z) = \sum_{k=0}^{\infty} C(k) z^{ak+b} \qquad a>0 \qquad b\geqslant 0 \qquad |z|<\infty \tag{3.2}$$

and $C(k)$ is non-alternating in sign as before. All we have done is suppress the alternating sign in equation (1.2), this being strongly correlated with the convergence of the original series (1.1) as illustrated by

$$\begin{aligned} f(s, \pm t) &\equiv \sum_m \lambda_m^{-s}\, \mathrm{e}^{\pm \lambda_m t} \qquad t>0 \\ &= \sum_m \lambda_m^{-s} \sum_{k=0}^{\infty} (\pm t)^k \frac{1}{k!} \lambda_m^k . \end{aligned} \tag{3.3}$$

For $+$ $(-)$ we have exponential divergence (convergence) of the left-hand series, and non-alternating (alternating) sign in $k$ on the right. Many other examples of the same correlation could be given. We choose equations (3.1), (3.2) as *representative* of divergent series $\tilde{F}(s, t)$ just as equations (1.1), (1.2) are representative of convergent series $F(s, t)$.

The derivation of equation (2.7) can now be repeated step by step, wth the result

$$\tilde{F}(s, t) = \sum_m \lambda_m^{-s} \tilde{f}(\lambda_m t)$$

$$= \sum_{k=0}^{\infty} C(k) t^{ak+b} Z(s - ak - b) + \{\,\}_p + \{\,\}_{ex} \tag{3.4}$$

where $\{\,\}_p$ and $\{\,\}_{ex}$ are given by equations (2.5) and (2.6) with the simple change cosec $z \to \cot z$. In equation (2.5*b*) the factor $(-)^L$ is deleted under the sum $\Sigma''$ as well. The linkage of $(-)^k$ with cosec $\pi z$ and $(+)^k$ with cot $\pi z$ is elementary and fully discussed in [4, 5, 7]. Equation (3.4) is, then, the value assigned by $\zeta$-function regularization to the divergent series $\tilde{F}(s, t)$. $\{\,\}_{ex}$ is exponentially small for small $t$, and disregarding this contribution, equation (3.4) becomes an asymptotic series for $\tilde{F}(s, t)$. The convergence or divergence of this asymptotic series can be investigated much as for the series (2.7), and no more will be said about this.

*Example.* In [5] a number of examples of equation (3.4) were given in which $\lambda_m = am + b$ is a linear function of $m = 1, 2, \ldots$ and $f(z)$ is cosh $z$, sinh $z$ or the corresponding Bessel functions $I_\nu(z)$, $L_\nu(z)$. One does not find such formulae, even in the largest collections of results on infinite series, the reason being of course that identities and not definitions belong in such collections. Let us consider here the series

$$\tilde{F}(s, t) = \sum_{m=1}^{\infty} m^{-s} e^{mt} \qquad t > 0$$

$$= \sum_{k=0}^{\infty} t^k \frac{1}{k!} \zeta(s-k) + \left\{ -\pi \cot \pi(s-1) \frac{t^{s-1}}{\Gamma(s)} \right\}_p . \tag{3.5}$$

Here $\{\,\}_{ex} = 0$. To gain some insight into why, let us set $x = 0$ in equation (3.5):

$$\tilde{F}(0, t) = \sum_{m=1}^{\infty} e^{mt}$$

$$= \sum_{k=0}^{\infty} t^k \frac{1}{k!} \zeta(-k) + \left\{ \frac{-1}{t} \right\}_p$$

$$= -\frac{1}{2} + \sum_{k=1}^{\infty} t^k \frac{1}{k!} \left[ -\frac{B_{k+1}}{k+1} \right] + \left\{ \frac{-1}{t} \right\}_p . \tag{3.6}$$

The final line may be compared with

$$\sum_{m=1}^{\infty} e^{mt} = -1 + (1 - e^t)^{-1} \tag{3.7}$$

which is the 'natural' value to assign to the left-hand series (e.g. see [8]). Recalling that

$$(e^t - 1)^{-1} = \sum_{n=0}^{\infty} t^{n-1} \frac{1}{n!} B_n \qquad 0 < |t| < 2\pi$$

is the generating function of the Bernoulli numbers $B_n$, we see that equations (3.6) and (3.7) are identical. Equation (3.5) is therefore the natural value to assign the divergent series $\tilde{F}(s, t)$ in the same sense that equation (3.7) is a natural definition.

## 4. Exponential series

Exponential series have particular importance, and this section will be devoted entirely to them. Known results on the heat kernel expansion will be compared with the general formulae derived in section 2, and complete agreement is found. This serves to confirm the exponential vanishing of $\{\,\}_{\mathrm{ex}}$ by a familiar and independent argument, and to draw attention to the geometrical significance of the $\zeta$-function $Z(s)$.

### 4.1. General formulae

For exponential series, one readily obtains, from the results of section 2,

$$f(s,t) \equiv \sum_m \lambda_m^{-s}\,\mathrm{e}^{-\lambda_m t}$$

$$= \sum_{k=0}^{\infty} (-t)^k \frac{1}{k!} Z(s-k) + \{\,\}_{\mathrm{p}} + \{\,\}_{\mathrm{ex}} \tag{4.1}$$

$$\{\,\}_{\mathrm{p}} = \sum_n t^{s-B+\Delta_n} R_n \Gamma(-s+B-\Delta_n) \tag{4.2}$$

$$\{\,\}_{\mathrm{ex}} = \frac{1}{2\pi \mathrm{i}} \int_C \mathrm{d}k\, \Gamma(-k) t^k Z(s-k). \tag{4.3}$$

To find these expressions we have used the identity $-\pi \operatorname{cosec} \pi k = \Gamma(-k)\Gamma(1+k)$. Pole cancellation occurs at points satisfying $s - B + \Delta_n = L = 0, 1, 2, \ldots$ and at these points one omits all singular terms from $\Sigma_k$ in equation (4.1), and replaces equation (4.2) by

$$\{\,\}_{\mathrm{p}} = \sum_n{}' \, t^{s-B+\Delta_n} R_n \Gamma(-s+B-\Delta_n) + \sum_n{}'' (-)^L \frac{t^L}{L!} [R_n(H_L - \gamma - \ln t) + C_n] \tag{4.4}$$

where $\Sigma''$ is over all poles for which $s - B + \Delta_n = L = 0, 1, 2, \ldots$. The constant $H_L = 1 + 1/2 + 1/3 + \ldots + 1/L$ comes from

$$\Gamma(-L+\varepsilon) = (-)^L \frac{1}{L!}\left(\frac{1}{\varepsilon} - \gamma + H_L + \mathrm{O}(\varepsilon)\right).$$

### 4.2. Heat kernel expansion

Consider an elliptic operator $A$ defined on a compact manifold $\mathcal{M}$ and having spectrum $\{\lambda_m\}$. Aside from depending on $A$, this spectrum also contains very detailed information about the spacetime manifold $\mathcal{M}$. How might one extract this information from the spectrum? The standard procedure is based on the asymptotic expansion (e.g. see [9])

$$f(0,t) = \sum_m \mathrm{e}^{-\lambda_m t} \sim \sum_{n=0}^{\infty} a_n t^{(n-N)/d} \tag{4.5}$$

where $N$ is the dimension of $\mathcal{M}$ and $d$ is the order of the operator $A$ (i.e. the mass dimension of the eigenvalues $\lambda_m$). The coefficients $a_n$ in equation (4.5)—often called the 'spectral invariants'—each contain a particular piece of geometrical information about the manifold $\mathcal{M}$. Usually $A$ is chosen to be the Laplacian, and then $a_0$ is proportional to the volume of $\mathcal{M}$, $a_1$ is proportional to the boundary surface of $\mathcal{M}$, and so on. The famous spectral geometry problem (e.g. see [13, 14]) is to determine how much of the geometry of $\mathcal{M}$ can be extracted from the spectrum $\{\lambda_m\}$ via the coefficients $a_n$.

Equations (4.1)-(4.4) with $s=0$ give the coefficients $a_n$ in terms of the $\zeta$-function $Z(s)$. These results have been known for a long time. Let us recall the standard approach to this problem (e.g. see [9]) which begins with the formula

$$\begin{aligned}\Gamma(s)Z(s) &= \Gamma(s)\sum_m \lambda_m^{-s} \\ &= \int_0^\infty \mathrm{d}t\, t^{s-1} \sum_m \mathrm{e}^{-\lambda_m t} \\ &= \int_0^1 \mathrm{d}t\, t^{s-1} \sum_m \mathrm{e}^{-\lambda_m t} + r(s)\end{aligned} \tag{4.6}$$

where $r(s)$ is the remaining integral over $1 \leqslant t \leqslant \infty$. Since only positive eigenvalues are included in equation (4.6), $r(s)$ is clearly non-singular throughout the finite $s$-plane. However, using equation (4.5) we find that the other part does have poles (remember, integration of asymptotic series is allowed):

$$\begin{aligned}\int_0^1 \mathrm{d}t\, t^{s-1} \sum_m \mathrm{e}^{-\lambda_m t} &\sim \int_0^1 \mathrm{d}t\, t^{s-1} \sum_{n=0}^{L} a_n t^{(n-N)/d} \\ &= \sum_{n=0}^{L} \frac{a_n}{s+(n-N)/d} \qquad \operatorname{Re} s > \frac{N-L}{d}\end{aligned} \tag{4.7}$$

where the integer $L>0$ is as large as desired. Equation (4.7) shows that $\Gamma(s)Z(s)$ in equation (4.6) can be continued to the left of $\operatorname{Re} s = B$ as far as desired, and only simple poles will be found, these being on the real axis at the evenly spaced points

$$s=(N-n)/d \qquad n=0,1,2,\ldots. \tag{4.8}$$

This argument identifies the $a_n$ as residues of the poles of $\Gamma(s)Z(s)$. If none of the points (4.8) coincide with the non-positive integers, then $\Gamma(s)$ is finite at these points, and each of them is a pole of $Z(s)$:

$$\begin{aligned} &Z\left(\frac{N-n}{d}+\varepsilon\right) = \frac{1}{\varepsilon}\frac{a_n}{\Gamma[(N-n)/d]} + \text{finite} \qquad n=0,1,2,\ldots \\ &\frac{N-n}{d} \neq 0,-1,-2,\ldots.\end{aligned} \tag{4.9}$$

However, if $(N-n)/d=-p$ where $p=0,1,2,\ldots$, then $\Gamma(s)$ has a pole at this point, and $Z(s)$ is finite there. This yields the special values

$$Z(-p) = (-)^p p!\, a_{N+pd} \qquad p=0,1,2,\ldots. \tag{4.10}$$

Equations (4.9) and (4.10) are important general statements. First, the poles of $Z(s)$ are specified, and the residues of these poles given in terms of the spectral invariants $a_n$. Second, in appropriate cases, special values of $Z(s)$ are obtained in terms of the $a_n$.

Let us compare the results just found with equations (4.1)–(4.3). For $s=0$, equation (4.1) becomes

$$\sum_m \mathrm{e}^{-\lambda_m t} = \sum_{k=0}^{\infty} (-t)^k \frac{1}{k!} Z(-k) + \left\{ \sum_{n \neq N+pd} R_n \Gamma\left(\frac{N-n}{d}\right) t^{(n-N)/d} \right\}_{\mathrm{p}} + \{\,\}_{\mathrm{ex}} \qquad p=0,1,2,\ldots . \tag{4.11}$$

Comparison of equations (4.11) and (4.5) yields $R_n\Gamma[(N-n)/d]=a_n$ for $n\neq N+dp$, $p=0,1,2,\ldots$ which is equation (4.9); and $(-)^k Z(-k)/k! = a_{N+dp}$, $p=0,1,2,\ldots$ which is equation (4.10).

We have reviewed such well-known material to make several points.

First, that the Mellin transform (4.6) plays quite an essential role in this standard derivation. As a rule, for summand functions $f$ in equation (1.1) other than the exponential, one does not have the luxury of such a formula. Our more general procedure in section 2 (which may be regarded as the inverse of the calculation (4.6)–(4.10)) does not favour any particular summand function, and works equally well for all of them.

The second point is that, according to the standard spectral geometry analysis based on equation (4.5), $\{\,\}_{\mathrm{ex}}$ in equation (4.3) does have the property we claim for this function in general, of being a function of $t$ beyond power series form.

Third, the exponential series or heat kernel function (4.5) is well suited for the spectral geometry programme, but other series could also be used for the same purpose. The really fundamental function in the entire analysis is not the heat kernel, but rather the $\zeta$-function $Z(s)$ defined by equation (1.3). It is $Z(s)$ which contains in purest form the geometrical information about the spacetime manifold one is attempting to extract. In the expansion (4.5) this information passes from $Z(s)$ into the spectral invariants $a_n$ via the relations (4.9), (4.10). Equation (2.7) does the same thing as efficiently for other summand functions $f(\lambda_m t)$. It seems, therefore, that in the problem of spectral geometry the traditional focus may be slightly off centre. One may as well study $Z(s)$ directly, without recourse to the heat kernel series (4.5), or to any other special case of equation (2.7).

## 5. $\eta$-function resummation

Given a spectrum $\{\lambda_m\}$ having both positive and negative eigenvalues, with $\lambda_m$ increasing without limit and sufficiently rapidly in either direction, one can construct from this spectrum an $\eta$-function:

$$H(s) \equiv \sum_m (\operatorname{sgn} \lambda_m)|\lambda_m|^{-s} \qquad \operatorname{Re} s > B'. \tag{5.1}$$

Such functions are related to $\zeta$-functions, but are potentially more complicated than the latter (e.g. see [9]). $H(s)$ has an abscissa of absolute convergence $\operatorname{Re} s = B' \geqslant 0$; a meromorphic continuation to the left of this abscissa; possible poles along the real axis for $s \leqslant B'$; and no pole at $s=0$. $H(s)$ possibly has *no* poles at all for $\operatorname{Re} s > -\infty$. This is the case for the simplest $\zeta$-functions:

$$\eta(s) \equiv \sum_{m=1}^{\infty} (-)^{m+1} m^{-s} = [1-2^{1-s}]\zeta(s) \tag{5.2}$$

$$\beta(s) \equiv \sum_{n=0}^{\infty} (-)^n (2n+1)^{-s}. \tag{5.3}$$

A general $\eta$-function (5.1) can be used to reorganize convergent series just as a $\zeta$-function can:

$$G(s,t) \equiv \sum_m (\operatorname{sgn} \lambda_m)|\lambda_m|^{-s} f(|\lambda_m| t)$$

$$= \sum_{k=0}^{\infty} (-)^k C(k) t^{ak+b} H(s-ak-b) + \{\,\}_{\mathrm{p}} + \{\,\}_{\mathrm{ex}} \tag{5.4}$$

where $f(z)$ is the function (1.2) and all steps are essentially the same as in section 2. The power function $\{\,\}_{\mathrm{p}}$ is given by a formula like equation (2.5), and for the same reasons. If $H(s)$ is an entire function then $\{\,\}_{\mathrm{p}}=0$. $\{\,\}_{\mathrm{ex}}$ is given by a formula like equation (2.6), and is exponentially small for $t \to 0$. Discarding $\{\,\}_{\mathrm{ex}}$ in equation (5.4) leaves an asymptotic series representation for $G(s, t)$.

Divergent series can also be reorganized with the help of $\eta$-functions. A simple example is

$$\sum_{m=1}^{\infty} (-)^{m+1} m^{-s} \,\mathrm{e}^{mt} = \sum_{k=0}^{\infty} t^k \frac{1}{k!} \eta(s-k) \tag{5.5}$$

which should be compared with equation (3.5). The parallel between $\zeta$-function resummation and $\eta$-function resummation is so close that it is hardly necessary to say more.

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## Appendix

For completeness, this appendix gives the Cauchy integral derivation of equation (2.7). This argument was formulated originally by Weldon [4], who derived in a simpler context the extra term $\{\,\}_{\mathrm{p}}$, but failed to notice the other function $\{\,\}_{\mathrm{ex}}$ generated by series commutation. The need for something beyond $\{\,\}_{\mathrm{p}}$ was observed by the author [5]. However, it was Elizalde and Romeo [7] who first noticed $\{\,\}_{\mathrm{ex}}$. This appendix extends the analysis of [4, 7] to a general spectrum, which is straightforward. Then, we concentrate on the exponentially small function $\{\}_{\rm ex}$, explaining why it has this behaviour in $t$, why it is an entire function of $s$, and so on. Previously these questions have been addressed only in the simpler context of [1].

*Series commutation problem*

Let us reconsider the series commutation problem in equation (2.1):

$$F(s,t)\equiv\sum_m \lambda_m^{-s}f(\lambda_m t)$$

$$=\sum_m \lambda_m^{-s}\sum_{k=0}^{\infty}(-)^k C(k)(\lambda_m t)^{ak+b} \tag{A1}$$

$$=\sum_m \lambda_m^{-s}\frac{1}{2\mathrm{i}}\oint_H \mathrm{d}k\,\operatorname{cosec}\pi k C(k)(\lambda_m t)^{ak+b} \tag{A2}$$

$$=\sum_m \lambda_m^{-s}\frac{1}{2\mathrm{i}}\oint_D \mathrm{d}k\,\operatorname{cosec}\pi k C(k)(\lambda_m t)^{ak+b} \tag{A3}$$

$$=\sum_m \lambda_m^{-s}\frac{1}{2\mathrm{i}}\int_{-\varepsilon+\mathrm{i}\infty}^{-\varepsilon-\mathrm{i}\infty} \mathrm{d}k\,\operatorname{cosec}\pi k C(k)(\lambda_m t)^{ak+b}+\Gamma(s,t) \tag{A4}$$

$$=\frac{1}{2\mathrm{i}}\int_{-\varepsilon+\mathrm{i}\infty}^{-\varepsilon-\mathrm{i}\infty} \mathrm{d}k\,\operatorname{cosec}\pi k C(k) t^{ak+b} Z(s-ak-b)$$

$$+\Gamma(s,t)\qquad \operatorname{Re} s>B+b+\varepsilon \tag{A5}$$

$$=\frac{1}{2\mathrm{i}}\oint_D \mathrm{d}k\,\operatorname{cosec}\pi k C(k) t^{ak+b} Z(s-ak-b)$$

$$+\{\}_{\rm ex}+\Gamma(s,t)\qquad \operatorname{Re} s>B+b+\varepsilon. \tag{A6}$$

One more step remains to be taken; but let us first explain the steps (A1)-(A6). In equation (A2), the contour $H$ is a counterclockwise hairpin enclosing the positive real $k$-axis. We assume that $C(k)$ has no singularities for $\operatorname{Re} k>-\infty$. Thus in equation (A2) the contour $H$ can be expanded until it becomes the perimeter $D$ of an infinite half-disc, whose flat side is $\operatorname{Re} s=-\varepsilon(\varepsilon\geqslant 0)$, enclosing the right half-plane in the counterclockwise sense. In equation (A4) the integral around $D$ is separated into the integral along the flat side, and the integral around the infinite semicircle $C$ of contour $D$:

$$\Gamma(s,t)\equiv\sum_m \lambda_m^{-s}\frac{1}{2\mathrm{i}}\int_C \mathrm{d}k\,\operatorname{cosec}\pi k C(k)(\lambda_m t)^{ak+b}. \tag{A7}$$

We shall assume that $\Gamma(s,t)$ vanishes identically, and this is a fairly weak assumption given reasonable convergence properties of the series (1.2). In equation (A5) the condition $\operatorname{Re} s>B+b+\varepsilon$ is maintained to keep $\Sigma_m$ convergent, so that $\Sigma_m$ and the integration along the flat side of contour $D$ can be commuted. In equation (A6) the contour $D$ is reclosed by adding and subtracting the contribution from the infinite semicircle $C$ of the half-disc perimeter $D$. Thus we arrive at the formal expression

$$\{\}_{\rm ex}=-\frac{1}{2\mathrm{i}}\int_C \mathrm{d}k\,\operatorname{cosec}\pi k C(k) t^{ak+b} Z(s-ak-b). \tag{A8}$$

The final step is to shrink the half-disc perimeter $D$ in equation (A6) back to the hairpin contour $H$ enclosing the positive real $k$-axis, and then re-express the integral around $H$ as a discrete sum:

$$F(s,t)=\sum_m \lambda_m^{-s} f(\lambda_m t)$$

$$=\sum_{k=0}^{\infty}(-)^k C(k) t^{ak+b} Z(s-ak-b)+\{\,\}_{\mathrm{p}}+\{\,\}_{\mathrm{ex}} \tag{A9}$$

where $\{\,\}_{\mathrm{ex}}$ is given by equation (A8). It remains to evaluate the power series extra term $\{\,\}_{\mathrm{p}}$, and this is easy. Weldon [4] noted that, when shrinking $D$ back to $H$ in equation (A6), any poles of $Z(s-ak-b)$ within $D$ have to be taken into account. It is easy to see that *all* of the poles of this $\zeta$-function lie within $D$ and therefore contribute, and altogether the contribution from all of these poles is precisely the function (2.5$a$). Remember that $\operatorname{Re} s > B+b+\varepsilon$ is still in force in equation (A6). The poles of $Z(s-ak-b)$ are located at $s-ak-b=B-\Delta_n$, $n=0,1,2,\ldots$ (see section 2.1) where $\Delta_n$ gives the separations between adjacent poles. Thus in the $k$-plane, the poles $k=(s-B-b+\Delta_n)/a$ all lie within $D$. Denoting the pole residues by $R_n$,

$$Z(B-\Delta_n+\varepsilon)=\frac{1}{\varepsilon}R_n+C_n+\mathrm{O}(\varepsilon) \tag{A10}$$

the contribution from the poles in $Z(s-ak-b)$ when going from equation (A6) to equation (A9) is, for $(s-B-b+\Delta_n)/a\neq 0,1,2,\ldots$, precisely $\{\,\}_{\mathrm{p}}$ in equation (2.5$a$). We assume that $C(z)$ is regular for $\operatorname{Re} s>-\infty$. This permits us to now relax the condition $\operatorname{Re} s>B+b+\varepsilon$, and continue the function (2.5$a$) throughout the $s$-plane. It has been assumed that $(s-B-b+\Delta_n)/a\neq 0,1,2,\ldots$ to keep the poles of $Z(s-ak-b)$ apart from the poles of $\operatorname{cosec}\pi k$. If $(s-B-b+\Delta_n)/a=L=0,1,2,\ldots$ for one or more poles of $Z(s-ak-b)$, then these poles sit atop poles of $\operatorname{cosec}\pi k$, and for $\{\,\}_{\mathrm{p}}$ one obtains precisely the function (2.5$b$).

### *Comments on* $\{\,\}_{ex}$

In the final result (A9), all terms on the right are known except for $\{\,\}_{\mathrm{ex}}$. The real complications of the series communication problem reside in this function, whose computation even in simple cases presents an extremely challenging problem. There are, however, some general statements which can be made about this function.

*Dependence on t.* Observe that $\{\,\}_{\mathrm{ex}}$ in equation (A8) vanishes as $t\to 0$, faster than any power of $t$. This is evident from the contour integral whose integrand is proportional to an infinite power of $t$ along the entire contour $C$ excepting the endpoints. In the class of series studied in [1], having $\lambda_m=m^\alpha$, $n=1,2,3,\ldots$. and $\alpha>0$, it was seen that 'vanishes faster than any power of $t$' means exponential behaviour $\exp(-\mathrm{const}/t)$. This is not yet proven in general for an arbitrary spectrum. However, it is probably true that, for an arbitrary spectrum, the function (A8) does vanish exponentially in the same way. We have no space here to fully discuss this. Let it suffice to say that if the high eigenvalues $\lambda_m$ with $m\to\infty$ have the leading behaviour $\lambda_m=a_0 m^\alpha+$smaller terms, then to leading order the same situation found in [1] will be reached. This comment can be expected into a substantial argument, if not a general proof.

*Dependence on s.* If the left-hand side of equation (A9) is an entire function of $s$, and on the right the term $\{\,\}_{\mathrm{p}}$ cancels all $\zeta$-function poles in the power series preceeding it, then $\{\,\}_{\mathrm{ex}}$ must also be an entire function of $s$. To what extent does equation (A8) agree with this? Suppose the $\zeta$-function $Z(s-ak-b)$ has only finitely many poles, at

$$s = ak + b + B - \Delta_n \qquad n = 0, 1, 2, \ldots, N. \tag{A11}$$

These equations map the $k$-plane contour $C$ in equation (A8) into a set of similar half-circle contours $C_n$, $n = 0, 1, 2, \ldots, N$ in the $s$-plane. However, all of these $s$-plane contours $C_n$ lie beyond finite $s$, at infinity. Thus, given the function (A8) is well defined, it will be entire in the $s$-plane, but singular along the curve at infinity $|s| = \infty$, $|\arg s| \leqslant \pi/2$. As $N$ grows arbitrarily large this situation persists: for any finite $\Delta_n$ no matter how large, the $s$-plane contours $C_n$ crowd together at infinity.

*Dependence on parameter a.* What will be described here is an (unproven) apparent scenario based on a number of examples. It is certainly true that $Z(s-ak-b)$ blows up on the contour $C$ in equation (A8). Our conjecture is that this occurs in such a way as to ensure the existence of a critical value $a_{\mathrm{c}}$ of parameter $a$ for which

$$\begin{aligned} \{\,\}_{\mathrm{ex}} &= 0 \qquad & 0 < a < a_{\mathrm{c}} \\ &\neq 0 \qquad & a \geqslant a_{\mathrm{c}} \end{aligned}$$

a possibility first pointed out by Elizalde and Romeo [7] in their work on series commutation in a specific example. In general, it is the blow up of $Z(s-ak-b)$ on contour $C$ that enables $\{\,\}_{\mathrm{ex}}$ to be non-vanishing. The rate at which $Z(s-ak-b)$ becomes infinite as this contour is approached is controlled by the positive parameter $a$. However, if we allow this parameter to decrease to $a = 0$, the $\zeta$-function no longer blows up on contour $C$, and $\{\,\}_{\mathrm{ex}}$ vanishes. Thus it seems inevitable that a critical value $a_{\mathrm{c}} > 0$ must exist as described above. Again we have no space to discuss this in any detail. However, a few remarks may be justified.

Any analysis of this question will be based on the rate at which $Z(s-ak-b)$ diverges as $\mathrm{Re}\, k \to \infty$. All $\zeta$-functions grow without bound far to the left in their complex plane. Analytic continuation of the defining series (1.3) is, of course, needed to reveal this. For the Riemann $\zeta$-function $\zeta(s)$, the reflection formulation (see [11] performs this analytic continuation very explicitly. For any more complicated and less fundamental $\zeta$-function, there does not exist such a formula. This is a great hindrance in many ways. In [1] we were above to make useful progress on understanding $\{\,\}_{\mathrm{ex}}$ for the class of series studied there (e.g. showing exponential vanishing as $t \to 0$) precisely because the Riemann $\zeta$-function was involved. Lacking such a convenient analytic continuation in the general case, one can at least examine some general classes of spectra. Consider, for example, spectra of the form

$$\lambda_m = m^{\alpha_1} + \beta_2 m^{\alpha_2} + \beta_3 m^{\alpha_3} + \ldots \qquad \alpha_1 > \alpha_2 > \alpha_3 + \ldots \tag{A12}$$

where at least $\alpha_1 > 0$. The upper end of the spectrum is dominated by $\lambda_m = m^{\alpha_1} + \ldots$ for $m \to \infty$, and $Z(s) = \zeta(\alpha_1 s) + \ldots$ where the corrections are proportional to $\zeta$-functions such as $\zeta[\alpha_1 s + n(\alpha_1 - \alpha_2)]$, $n = 1, 2, \ldots$ evaluated further to the right in the complex plane. Consequently, $Z(s)$ is dominated by $\zeta(\alpha_1 s)$ for $\mathrm{Re}\, s \to -\infty$. Then one can use the reflection formula for $\zeta[\alpha_1(s-ak-b)]$ to find the leading behaviour of $Z(s-ak-b)$ as $\mathrm{Re}\, k \to -\infty$, precisely as was done in [1]. In this fashion one recovers the exponential vanishing of $\{\,\}_{\mathrm{ex}}$ with $t$, and other general features of the analysis in [1], including the existence of the critical parameter $a_{\mathrm{c}} > 0$. We plan to amplify these brief remarks elsewhere.

## References


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